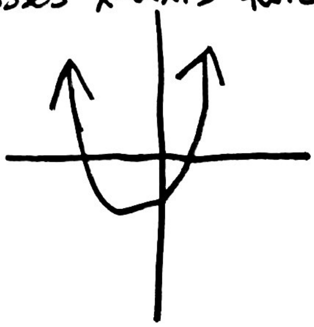
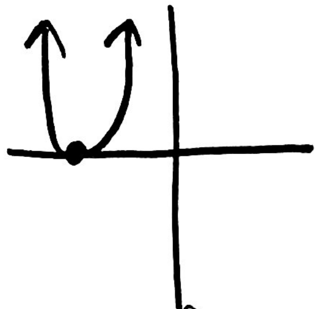
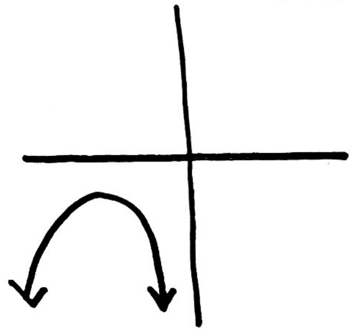
Unit 3 Solving Polynomials

A *Polynomial Equation* contains both a polynomial and an equal sign. The solution to a polynomial equation is called a *root*. If a polynomial equation has a zero on one side of the equal sign, then the roots are called *zeros* of the polynomial (because they are values that make the polynomial equal 0). Throughout this unit, we will find out how to solve polynomial equations. The terms solution, root, and zero will be used largely interchangeably.

3.1 Theorems of Polynomials

Fundamental Theorem of Algebra, Zero Product Property, Rational Root Theorem, Remainder Theorem

Objective: Use theorems of Polynomials to identify the number of solutions to a polynomial equation and identify what those solutions might be.

Solutions of Quadratics		
Two Real Zeros	One Real Zero, multiplicity 2	No Real Zeros
Crosses x-axis twice	Hits x-axis once	Doesn't touch x-axis
		
Ex:	Ex: $y = (x+3)^2$	Ex:

*Multiplicity 2 since the zero happens twice

Every quadratic polynomial equation has two roots. Every cubic polynomial equation has three roots, and so on. This result is related to the **FUNDAMENTAL THEOREM OF ALGEBRA.**

FUNDAMENTAL THEOREM OF ALGEBRA:
If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x)=0$ has exactly n roots, including multiple roots and complex roots.

Meaning: The degree of a polynomial says how many solutions there are (real, repeated, and imaginary)

This means that the degree of a polynomial in an equation tells you exactly how many roots (solutions) that equation has, whether they be rational, irrational, or imaginary.

Example 1. How many roots does each of the following polynomial equations have?

a) $2x^4 - x^3 + 3x^2 - 1 = 0$ b) $9x^{15} - 4x^6 + 10 = 0$

4 solutions 15 solutions

Words that mean the same thing:

Roots
Solutions
Zeros

Previously, we have practiced factoring polynomials into linear factors. This wasn't just for fun. We can use the factored form of a polynomial to easily find the roots of a polynomial equation. We will use the **ZERO PRODUCT PROPERTY** to our advantage here.

ZERO PRODUCT PROPERTY:
Any value of x that makes one factor of a polynomial equal zero will make the entire polynomial equal zero.

To use the Zero Product Property, we get a zero on one side of the polynomial equation and then completely factor the other side. Once we are down to the linear factors, set each linear factor equal to zero and solve for x. There will be one solution for each linear factor, just like the fundamental theorem of algebra says.

Example 2. Find all solutions for each polynomial equation. Then state the degree.

a) $(x-4)(x+8)(x-2)=0$

$\boxed{x=4}$ $\boxed{x=-8}$ $\boxed{x=2}$

Degree 3

b) $(x+3)(x-6)^2(5x-3)=0$

$\boxed{x=-3}$ $\boxed{x=6 \text{ mult. } 2}$ $\boxed{x=\frac{3}{5}}$

Degree 4

c) $x(x+6)(x-5)^3=0$

$\boxed{x=0}$ $\boxed{x=-6}$ $\boxed{x=5 \text{ mult. } 3}$

Degree 5

d) $x^2(3x+4)(5x-8)(x+6)=0$

If you want to simply know if a given number is a zero of a polynomial equation, you could use the **REMAINDER THEOREM.**

THE REMAINDER THEOREM
If you divide polynomial $P(x)$ ~~of degree~~ $n \geq 1$ by $x-a$, then the remainder is $P(a)$.

Meaning: If you divide a polynomial by a factor, the remainder is the same as if you plugged in the zero of the factor

Example 3. Use synthetic division and the Remainder Theorem to find $P(a)$.

a) $P(x)=6x^3-x^2+4x+3; a=3$

$$\begin{array}{r|rrrr} 3 & 6 & -1 & 4 & 3 \\ & & 18 & 51 & 165 \\ \hline & 6 & 17 & 55 & \boxed{168} \end{array}$$

vs. $6(3)^3-(3)^2+4(3)+3$

$=\boxed{168}$

b) $P(x)=2x^3-x^2+10x+5; a=\frac{1}{2}$

$2(\frac{1}{2})^3-(\frac{1}{2})^2+10(\frac{1}{2})+5$

$=\boxed{10}$

The reason that this is important is that we can use the remainder theorem to see if $P(a)=0$, if it does, then a is a zero of the polynomial—one of the solutions that we are looking for.

Remainder of 0 says the value is a zero/root/solution.

Example 4. Use synthetic division and the remainder theorem to determine whether or not each value of a is a zero of the polynomial $P(x)=x^3+2x^2-40x+64$

a) $a=6$

b) $a=4$

Now that we have a way to check if a number is a root of a polynomial equation, it would be great if we had a way to narrow down the list of possible roots, so we only have to check a few of them. This brings us to the

RATIONAL ROOT THEOREM

Before we dive into that, though, just a refresher on types of numbers. Rational numbers can be written as fractions (sometimes there is just a 1 in the denominator) and can be written as either a terminating or repeating decimal.

RATIONAL ROOT THEOREM

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$ be a polynomial with integer coefficients. There are a limited number of possible roots or zeros of $P(x) = 0$:

- Integer roots must be factors of a_0.
- Rational roots must have reduced form $\frac{p}{q}$ where p is an integer factor of a_0 and q is an integer factor of a_n.

Meaning: To get a list of all potential rational zeros of a polynomial:

$$\pm \frac{\text{Factors of constant (last number)}}{\text{Factors of first coefficient}}$$

The Rational Root Theorem lets us generate a list of possible rational roots for a polynomial, based on the integer factors of the leading coefficient and the integer factors of the constant.

Example 5. What are the possible rational roots for the following polynomials:

a) $3x^3 + 7x^2 + 6x - 8 = 0$

b) $6x^3 + 2x - 18 = 0$

Last(8): 1, 2, 4, 8

First(3): 1, 3

Find all possible numbers that multiply to your last number (constant) and your first coefficient.

$\pm\frac{1}{1}, \pm\frac{1}{3}, \pm\frac{2}{1}, \pm\frac{2}{3}, \pm\frac{4}{1}, \pm\frac{4}{3}, \pm\frac{8}{1}, \pm\frac{8}{3}$

$= \boxed{\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}, \pm 8, \pm\frac{8}{3}}$

Each factor of the constant gets a turn being on top of each factor of the first coefficient.

By combining the Rational Root Theorem and the Remainder Theorem, we have a method for determining rational roots of polynomial equations. Use the Rational Root Theorem to list possible rational roots, then use the Remainder Theorem to test those roots and see which ones work.

Example 6. List all possible roots for each polynomial equation, then find any actual roots.

a) $x^3 + 5x^2 + x + 5 = 0$

b) $3x^4 + 2x^3 - 9x^2 + 4 = 0$

Last (5): 1, 5

First (1): 1

$\pm\frac{1}{1}, \pm\frac{5}{1}$

$= \pm 1, \pm 5$

Test each possibility with the remainder theorem or synthetic division.

$(1)^3+5(1)^2+(1)+5 = 12$

$(-1)^3+5(-1)^2+(-1)+5 = -2$

$(5)^3+5(5)^2+(5)+5 = 260$

$(-5)^3+5(-5)^2+(5)+5 = 0$

$\boxed{-5 \text{ is a root}}$

(since there was a remainder of 0)

c) $2x^3 + 7x^2 - 5x - 4 = 0$

Last (4): 1, 2, 4

First (2): 1, 2

$\pm\frac{1}{1}, \pm\frac{1}{2}, \boxed{\pm\frac{2}{1}}, \pm\frac{2}{2}, \pm\frac{4}{1}, \boxed{\pm\frac{4}{2}}$

$= \pm 1, \pm\frac{1}{2}, \pm 2, \pm 4$

* Don't need to write numbers more than once

$2(1)^3+7(1)^2-5(1)-4 = \boxed{0}$

$2(-1)^3+7(-1)^2-5(-1)-4 = 6$

$2(\frac{1}{2})^3+7(\frac{1}{2})^2-5(\frac{1}{2})-4 = -4.5$

$2(-\frac{1}{2})^3+7(-\frac{1}{2})^2-5(-\frac{1}{2})-4 = \boxed{0}$

$2(2)^3+7(2)^2-5(2)-4 = 30$

$2(-2)^3+7(-2)^2-5(-2)-4 = 18$

$2(4)^3+7(4)^2-5(4)-4 = 216$

$2(-4)^3+7(-4)^2-5(-4)-4 = \boxed{0}$

Suggestion: Plug equation into calculator and use $\boxed{\text{Vars}} \rightarrow$ Y-Vars to calculate all of the values

$\boxed{1, -\frac{1}{2}, \text{ and } -4 \text{ are roots of the polynomial}}$

3.2 Solving Quadratics

Objective: Identify types of solutions to a quadratic equation and use a variety of methods to solve quadratic equations and related cubics.

A quadratic equation is a second degree polynomial equation. Any time we solve a quadratic equation, just like any other polynomial equation, our first step is to get a zero on one side of the equation and the polynomial on the other. The resulting form should look something like this:

$$ax^2 + bx + c = 0$$

Where a, b, and c are integers (and a should be positive).

When solving quadratic equations, a good place to start is to determine what type of solutions you will get. When solving a quadratic, solutions will either be rational, irrational, or imaginary. A small part of the quadratic formula will tell us what our solutions will look like. This part of the quadratic formula is called the *discriminant*.

The discriminant is b^2-4ac.

So basically the number underneath the square root in the quadratic formula determines what type of solution you'll have.

Discriminant	Solution Type
Negataive	Two imaginary
Positive (not a perfect square)	Two irrational
Positive (perfect square)	Two rational
Zero	One repeated

Example 1. Use the discriminant to find what type of solutions—and how many—each quadratic will have.

a) $7x^2 - 4x - 3 = 0$

$a=7 \quad b=-4 \quad c=-3$

$(-4)^2-4(7)(-3)=100$

Perfect square

$\boxed{\text{Two rational}}$

b) $10x^2 + 5x + 3 = 0$

$a=10 \quad b=5 \quad c=3$

$(5)^2-4(10)(3)=-95$

$\boxed{\text{Two imaginary}}$

c) $5x^2 - 9x + 4 = 0$

$a=5 \quad b=-9 \quad c=4$

$(9)^2-4(5)(4)=-44$

$\boxed{\text{Two imaginary}}$

d) $2x^2 + 4x + 2 = 0$

$a=2 \quad b=4 \quad c=2$

$(4)^2-4(2)(2)=0$

$\boxed{\text{One repeated}}$

What are the strategies that you have learned to solve quadratics?

Factoring & Quadratic Formula

Which strategy is easier for you?

Which strategy will always work?

Quadratic Formula: $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

* Make sure equations are set equal to 0

Example 2. Solve each of the following quadratics.

a) $4x^2 - 2x - 2 = 0$

$2(2x^2 - x - 1) = 0$

$-2x^2$

$-2x \quad 1x$

$2(2x^2 - 2x + 1x - 1) = 0$

$2(2x(x-1) + 1(x-1)) = 0$

$2(x-1)(2x+1)$

$\boxed{x=1}$ $\boxed{x=-\frac{1}{2}}$

b) $4x^2 - 5x + 4 = 0$

$a=4 \quad b=-5 \quad c=4$

$x = \dfrac{5 \pm \sqrt{(-5)^2 - 4(4)(4)}}{2(4)}$

$= \dfrac{5 \pm \sqrt{-39}}{8} = \boxed{\dfrac{5 \pm i\sqrt{39}}{8}}$

c) $x^2 + 2x = -1$

d) $2x^2 + 2x - 4 = 0$

e) $3x^2 = x + 2$

f) $2x^2 + 2x + 2 = 0$

Some equations lend themselves to shortcuts that allow us to solve them even faster. Let's explore a few special cases to factoring:

Difference of Squares:	$a^2 - b^2 = (a+b)(a-b)$	Basically take square root of both terms

Example 3. Solve each of the following polynomial equations by factoring.

a) $x^2 - 16 = 0$

$(x+4)(x-4)$

$\boxed{x=-4}$ $\boxed{x=4}$

b) $5x^3 - 45x = 0$

$5x(x^2-9) = 5x(x+3)(x-3)$

$\boxed{x=0}$ $\boxed{x=-3}$ $\boxed{x=3}$

c) $x^2 - 10 = 0$

$(x+\sqrt{10})(x-\sqrt{10})$

$\boxed{x=-\sqrt{10}}$ $\boxed{x=\sqrt{10}}$

d) $x^3 - 2x^2 + x = 0$

$x(x^2 - 2x + 1) = 0$

$1x^2$

$-1x \quad -1x$

$x(x-1)(x-1)$

$x=0 \quad x=1 \quad x=1$

$\boxed{\begin{array}{l} x=0 \\ x=1 \text{ mult. } 2 \end{array}}$

There's one other shortcut that we can use now that we've had some experience with imaginary roots of polynomial equations. We know that $x^2 - 9$ factors out to be $(x+3)(x-3)$. What happens when we multiply $(x+3i)(x-3i)$ together? Try it now and see what you get?

This leads us to a shortcut called the *Sum of Squares*. It works a lot like the Difference of Squares, but we put an i with the second term of each factor.

Sum of Squares:

$$a^2 + b^2 = (a+bi)(a-bi)$$

Example 4. Use the Sum of Squares rule to solve each of the following quadratic equations.

a) $x^2 + 25 = 0$

*Create a difference by rewriting with a negative

$x^2 - (-25) = 0$

$\sqrt{-25} = 5i$

$(x+5i)(x-5i) = 0$

$\boxed{x=-5i}$ $\boxed{x=5i}$

b) $16x^2 + 49 = 0$

$16x^2 - (-49) = 0$

$\sqrt{16} = 4$

$\sqrt{-49} = 7i$

$(4x+7i)(4x-7i) = 0$

$\boxed{x = \frac{-7i}{4}}$ $\boxed{x = \frac{7i}{4}}$

c) $x^2 + 36 = 0$

$\sqrt{-36} = 6i$

$x^2 - (-36) = 0$

$(x+6i)(x-6i)$

$\boxed{x=-6i}$ $\boxed{x=6i}$

d) $9x^3 + 81x = 0$

$9x(x^2+9) = 0$

$\sqrt{-9} = 3i$

$9x(x^2-(-9)) = 0$

$9x(x+3i)(x-3i) = 0$

$\boxed{x=0}$ $\boxed{x=-3i}$ $\boxed{x=3i}$