3.3 Building Polynomials From Roots and the Conjugate Theorem

Objective: Build up a polynomial given all of the roots or just some of them.

So far, we have solved polynomial equations by factoring the polynomials down to their linear factors and then using the Zero Product Property to get the roots of the polynomial from those linear factors. It is very important that we understand the difference between a factor and a root.

For the polynomial equation $x^2 + 6x - 16 = 0$, it is easy enough to factor the left side of the equation and come up with $(x+8)(x-2) = 0$. In this form $(x+8)$ and $(x-2)$ are both factors of the polynomial. We use the factors to find the roots of the polynomial—which in this case would be -8 and 2. Roots are sometimes referred to as solutions or zeros, but those mean basically the same thing. Factors, though, are different from roots.

In this section we will be given a list of roots from which we will build a polynomial expression. Sometimes this list is complete and sometimes we'll need to fill in some blanks.

If all of the numbers in our list of roots are rational, then we can assume that the list is complete and we just need to use the numbers we're given. Start by writing out a factor for each root, $(x-r)$ where r is the given root. Then multiply the factors together to get the polynomial in standard form.

Roots	Factors	Polynomial
2, -3, 4 $x=2$ $x=-3$ $x=4$	$(x-2)$ $(x+3)$ $(x-4)$	Multiply all factors together $(x-2)(x+3)(x-4)$ $(x^2+x-6)(x-4)$ $\boxed{f(x) = x^3-3x^2-10x+24}$

Example 1. Write a polynomial equation of least degree that has the given roots.

a) 2, 3, 4

b) -5, 6, -7

c) 8, $\frac{1}{9}$

$x=8$

$x=\frac{1}{9}$

$(x-8)(9x-1)$

$9x^2 - x$
$-72x + 8$

$\boxed{f(x) = 9x^2 - 73x + 8}$

d) -2, 2, -3, 4

If the given roots are not all rational—i.e. there is an irrational or imaginary root given—then we need to fill in some blanks. For each irrational or imaginary root of a polynomial, the *conjugate* must also be a root of the polynomial. If you look at the quadratic formula, you can see why this would be the case.

Conjugate:
If a number can be written as $a + b$, then the conjugate of that number is $a - b$.

Example 2. Write the conjugate of each number below.

* Answers come in conjugate pairs

* Conjugates happen with $\sqrt{\ }$ or i

a) $3 - 4i$

$3+4i$

b) $6 + 2\sqrt{3}$

$6-2\sqrt{3}$

c) $7i$

$-7i$

d) $-5\sqrt{2}$

$5\sqrt{2}$

e) $8 + 10i$

$8-10i$

f) $10 - \sqrt{7}$

$10+\sqrt{7}$

Now that we know what a conjugate is, we can try building polynomials that contain irrational and imaginary roots.

Key idea: Work backwards to get your polynomial

Example 3. Write a polynomial equation, with rational coefficients, of least degree that has the given roots.

a) $\sqrt{14}$

$x^2=(\sqrt{14})^2$

$x^2=14$

$x^2-14=0$

$\boxed{f(x)=x^2-14}$

b) $2i$

c) $3 - 4i$

d) $-\sqrt{5}$

$x^2=(-\sqrt{5})^2$

$x^2=5$

$x^2-5=0$

$\boxed{f(x)=x^2-5}$

Square both sides to get rid of $\sqrt{\ }$

e) $8i$ * $x^2=(8i)^2$

$x^2=-64$

$x^2+64=0$

$\boxed{f(x)=x^2+64}$

f) $9 + \sqrt{5}$

$x=9+\sqrt{5}$

$x-9=\sqrt{5}$

$(x-9)^2=(\sqrt{5})^2$

$(x-9)(x-9)=5$

$x^2-18x+81=5$

$x^2-18x+76=0$

$\boxed{f(x)=x^2-18x+76}$

* i comes from negative under $\sqrt{\ }$, so square both sides

Now we can use everything we know about roots and conjugates to tackle these problems. Here's a **pro tip**: if you've got 2 roots that are conjugates of one another, it is easiest to multiply the factors from those conjugates together first, then distribute the other factors through your polynomial.

Example 4. Write a polynomial equation, with rational coefficients, of least degree that has the given roots.

a) $3, 8, \sqrt{2}$

$(x-3)(x-8)$ $x=\sqrt{2}$

$x^2=2$

x^2-2

$f(x)=(x-3)(x-8)(x^2-2)$

$=(x^2-11x+24)(x^2-2)$

$\boxed{f(x)=x^4-11x^3+22x^2+22x-48}$

b) $4\sqrt{2}, -6, -9\sqrt{3}$

c) $4, -6, -9i$

$(x^2-2)(x^2-11x+24)$

$x^4-11x^3+24x^2$

$-2x^2+22x-48$

d) $5, 10i, 4 - i$

$(x-5)$ $x=10i$

$x^2=-100$

x^2+100

$x=4-i$

$x-4=-i$

$(x-4)^2=(-i)^2$

$x^2-8x+16=-1$

$x^2-8x+17$

$f(x)=(x-5)(x^2+100)(x^2-8x+17)$

$=(x^3-5x^2+100x-500)(x^2-8x+17)$

$\boxed{f(x)=x^5-13x^4+157x^3-1385x^2+5700x-8500}$